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Critical behaviour of the long-range $(\phi^2)^2$ model in the short-range limit

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Abstract. The relation between scaling regimes described by $(\phi^2)^2$ field theories with long-range and short-range exchange is analysed. It is shown that the crossover from the long-range regime to the short-range regime takes place due to the singular behaviour of three-loop and higher-order contributions to the beta function of the renormalisation group, which controls the scaling behaviour of the long-range model. The critical exponents η and ν are shown to be continuous functions of the parameter α, which characterises the power-like fall-off of the exchange interaction.

1. Introduction

The critical behaviour of a physical system is often described by models, which differ only in the range of the effective interactions: they are local in one case, and have a long range in the other. The question about the relation of the two descriptions has been investigated on several occasions [1-5]. In this paper, we consider the simplest model for which these problems have been studied: the $O(n)$ symmetric $(\phi^2)^2$ field theory with both local and long-range exchange terms (n is the number of components of the field ϕ), whose basic action we write in the form

$$S = -\tfrac{1}{2}a\nabla\phi\nabla\phi - \tfrac{1}{2}\tau\phi^2 - \tfrac{1}{2}b\phi(-\nabla^2)^{1-\alpha}\phi - \tfrac{1}{24}\lambda(\phi^2)^2 \tag{1}$$

where the values of parameters $a>0$, $\tau \geqslant 0$ and $\lambda>0$ correspond to the symmetric phase with zero expectation value of the field ϕ. In (1) and all subsequent similar formulae necessary integrals and sums are implied. The non-integer power of the Laplace operator is defined through the Fourier transform (the same notation is used for the field and its Fourier transform)

$$\int \mathrm{d}\boldsymbol{x}\,\phi(\boldsymbol{x})(-\nabla^2)^{1-\alpha}\phi(\boldsymbol{x}) = \int \frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^d}\,\phi(-\boldsymbol{p})\boldsymbol{p}^{2(1-\alpha)}\phi(\boldsymbol{p})$$

where d is the dimension of space. The original problem considered in the early work of Sak [1] was that of the superficial discontinuity of anomalous dimensions of field theories with short-range (to which corresponds $b=0$ in the action (1)) and long-range ($a=0$) exchange terms. For the short-range model, the renormalisation of the exchange term in the action gives rise to a non-trivial anomalous dimension γ_ϕ of the field ϕ (or, equivalently, non-zero critical exponent $\eta=\gamma_\phi$), whereas in the long-range case the exchange term is not renormalised at all, the anomalous dimension of the field is zero and the exponent η is equal to $\eta=2\alpha$. In the formal limit $\alpha\to 0$ the expressions for η (and other critical exponents as well) do not coincide. However, it was shown by Sak [1] at the leading non-trivial order of both $\varepsilon = 4 - d$ and $\varepsilon = d - 2$ expansions that the anomalous dimension of the field γ_ϕ and the other exponents, in fact, are continuous functions of the parameter α in the sense that the scaling regime described by the long-range field theory is valid for $\alpha > \eta/2$, whereas for $\alpha < \eta/2$ the scaling behaviour is described by the short-range model, and at the borderline value $\alpha = \eta/2$ the two descriptions yield equal values for the exponents.

This calculation was done in a scheme where both exchange terms of the action (1) were treated on an equal footing, giving rise to a double expansion in α and ε instead of the usual ε expansion. Similar techniques were later used in the analysis of other models with competing short-range and long-range behaviour [2-4], and recently the conjecture of Sak was generalised to all orders in perturbation theory [5]. This approach accounts for the crossover from scaling behaviour described by the short-range model to that described by the long-range model. However, to obtain full consistency in the theory, there should be a description of the crossover in the opposite direction, i.e. from the long-range regime to the short-range one. An attempt has been made to analyse this crossover [5], in which the short-range term $\nabla\phi\nabla\phi$ was treated as a 'dangerous' irrelevant operator. The stability of scaling behaviour with respect to perturbation by such field operators can be analysed in a fairly standard way [6]. This amounts to calculating the anomalous dimension of the corresponding composite operator (the naive scaling dimension of which suggests it to be an irrelevant operator), and checking whether or not the total scaling dimension (naive + anomalous) renders the operator relevant. In some cases, this scheme works neatly [4]. In the case of the long-range $(\phi^2)^2$ model, however, the anomalous dimension of the operator $\nabla\phi\nabla\phi$ turned out to be positive in the leading order of the ε expansion, and therefore in this scheme there was no sign of the long-range scaling regime becoming unstable at $\alpha \to 0$. The purpose of this paper is to complete the analysis of [5] in this respect: it is shown that the long-range scaling regime becomes unstable at $\alpha = \alpha^* = (n+2)\varepsilon^2/4(n+8)^2$, where $\varepsilon = 4 - d$, and the critical exponents η and ν assume their short-range values for $\alpha \leqslant \alpha^*$, so that the picture of the crossover between the scaling regimes described by the $(\phi^2)^2$ model is indeed also consistent in this respect.

2. Renormalisation of the long-range model

We start with the basic action of the field theory, which describes the long-range scaling regime

$$S_{\rm LR} = -\tfrac{1}{2}\phi(-\nabla^2)^{1-\alpha}\phi - \tfrac{1}{24}\lambda\mu^{\varepsilon}(\phi^2)^2 \qquad (2)$$

where μ is the scale-setting parameter and $\varepsilon = d_c - d$. The upper critical dimension d_c (at which the field theory is logarithmic) is equal to $d_c = 4 - 4\alpha$. Power counting shows that this model is multiplicatively renormalisable, and since the exchange term is non-analytic at the origin as a function of momentum $\boldsymbol{p}$, it is subject to finite renormalisation only. Indeed, all the self-energy graphs of the long-range theory have (at $d = d_c$) a degree of divergence δ equal to $\delta = 2 - 2\alpha$, which formally corresponds to divergent graphs. However, differentiating these graphs twice with respect to the external momentum, we obtain graphs with $\delta = -2\alpha$, which are convergent at large momenta after the subtraction of divergences corresponding to subgraphs of the original graphs. Therefore, the differentiated graphs do not produce any divergent contributions to the self-energy renormalisation constants. Since in this procedure we may have lost at most a constant term, it follows, in particular, that the field renormalisation constant (i.e. that of the gradient term in the long-range action (2)) is a finite quantity.

For $\alpha \gg \varepsilon = 4 - 4\alpha - d$ the short-range term $\nabla\phi\nabla\phi$ is obviously irrelevant, whereas for α small enough the canonical dimensions of the short-range and long-range terms become comparable and their interplay should be taken into account. For this, two different approaches have been suggested: first, it is possible to construct a double expansion in ε and α near the upper critical dimension $d_c = 4$ and analyse the stability of the two non-trivial fixed points appearing in this scheme [1, 5]. The long-range term must be treated as a perturbation to the short-range one in this method. Therefore it is suitable for description of the crossover from the scaling regime described by the short-range fixed point to the regime described by the long-range fixed point. Second, the analysis of the opposite case, in which the long-range model (2) is perturbed by the short-range term, at first seems feasible [5] in the standard approach based on composite field operator renormalisation [6]. The idea of the method is as follows: consider the relative behaviour of the Green functions of the field theory (2) with and without an insertion of the operator $O_2 \equiv \nabla_\phi\nabla_\phi$ (we remind ourselves that this definition implicitly includes an integral over the coordinate space and a sum over the field components), i.e. take a field theory with the basic action

$$S = -\tfrac{1}{2}\phi(-\nabla^2)^{1-\alpha}\phi - \tfrac{1}{24}\lambda\mu^\varepsilon(\phi^2)^2 - \tfrac{1}{2}t\nabla\phi\nabla\phi \tag{3}$$

where the short-range term has been introduced, and expand all the Green functions to the lowest order in the 'coupling constant' t. For instance, for the inverse full propagator $\Gamma_{\phi\phi}$ of the field theory (3) we obtain

$$\Gamma_{\phi\phi} = \Gamma^{\mathrm{LR}}_{\phi\phi} + t\Gamma^{\mathrm{LR}}_{\phi\phi,O_2} + \dots \tag{4}$$

where by LR we denote one-particle irreducible (1PI) Green functions of the long-range model (2). The difference between the scaling dimensions of the Green functions on the right-hand side of the relation (4) is, by definition, the dimension of the composite operator $d_{O_2} = d_{\Gamma_{\phi\phi,O_2}} - d_{\Gamma_{\phi\phi}}$ and without the loop correlations we obtain $d_{O_2} = 2\alpha > 0$, which indicates that in the relation (4) the Green function with the O_2 insertion in (4) is negligible in the long-distance limit.

Surprisingly, this conclusion does not change when the one-loop contributions are taken into account. The renormalisation constants of the interaction vertex Z_1 and the composite operator Z_2 are introduced in the following fashion:

$$S_{\mathrm{R}} = -\tfrac{1}{2}\phi(-\nabla^2)^{1-\alpha}\phi - \tfrac{1}{24}\lambda\mu^\varepsilon Z_1 C_d^{-1}(\phi^2)^2 - \tfrac{1}{2}tZ_2\nabla\phi\nabla\phi$$

where the factor $C_d^{-1} = 2^{d-1}\pi^{d/2}\Gamma(d/2)$ (Γ is the gamma function) has been introduced for convenience. We use dimensional regularisation with minimal subtractions, and at one-loop order obtain the familiar expression for the beta function:

$$\beta_\lambda = \lambda\left(-\varepsilon + \frac{n+8}{6}\lambda\right). \tag{5}$$

The value of the function $\gamma_2(\lambda) = -\mu(\partial \ln Z_2/\partial\mu)|_0$, where the subscript denotes that the derivative is taken with fixed value of the bare coupling constant λ_0, at the infrared stable fixed point $\lambda = \lambda^*$ of the renormalisation group determines the anomalous dimension of the composite operator O_2. At one-loop level the total scaling dimension of the operator O_2 is given by

$$d_{O_2} = 2\alpha + \frac{3(n+2)(1-2\alpha)}{2(n+8)^2(1-\alpha)}\varepsilon^2$$

which is positive for small positive α and thus does not signal the long-range regime becoming unstable at any $\alpha > 0$.

3. Stability of the long-range fixed point

There is, however, another potential source of instability in the long-range field theory: at small α it is not sufficient to use the first-order expression (5) for the beta function to obtain the correct value of the fixed point and to judge its stability. The reason is that the self-energy graphs of the long-range model, which are finite for finite $\alpha > 0$, become actually divergent in the limit $\alpha \to 0$. These divergences show in the form of poles $1/\alpha$, and the higher-order contributions to the beta function, which contain such singularities, should be taken into account for small α. The first correction of this type comes from the three-loop vertex graph shown in figure 1, which yields

$$Z_4 = 1 + \frac{n+8}{6}\frac{\lambda}{\varepsilon} - \frac{(n+8)(n+2)\Gamma^2(2-2\alpha)\Gamma(1+\alpha)}{648\alpha(1-\alpha)\Gamma(3-3\alpha)}\frac{\lambda^3}{\varepsilon} + \dots .$$

To the leading order in α, this results in the following modification of the beta function:

$$\beta_\lambda = \lambda\left(-\varepsilon + \frac{n+8}{6}\lambda - \frac{(n+8)(n+2)}{432}\frac{\lambda^3}{\alpha}\right).$$

Note the sign of the last term, which already indicates that the fixed point becomes unstable at $\alpha \to 0$. More precisely, the stability of the fixed point is determined by the value of the derivative of the beta function at the fixed point, for which we obtain (assuming $\alpha \propto \varepsilon^2$)

$$\left.\frac{\mathrm{d}\beta_\lambda}{\mathrm{d}\lambda}\right|_{\lambda=\lambda^*} = \varepsilon\left(1 - \frac{(n+2)}{(n+8)^2}\frac{\varepsilon^2}{\alpha}\right).$$

The fixed point is infrared stable when this quantity is positive. Therefore we see that the fixed point, which controls the scaling behaviour of the long-range model, becomes unstable at

$$\alpha = \alpha^* = \frac{n+2}{(n+8)^2}\varepsilon^2. \tag{6}$$

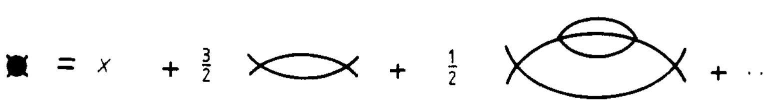

Figure 1. First relevant terms in the graphical expansion of the full four-point 1PI Green function (full circle on the left-hand side) of the long-range field theory (2). Lines of the graphs correspond to the bare propagators $G_0(\boldsymbol{p}) = 1/\boldsymbol{p}^{2(1-\alpha)}$, and vertices to the factor $-\lambda\mu^\varepsilon$ of the interaction vertex of the basic action (2). The pertinent symmetrisation of the graphs with respect to external arguments is not shown explicitly, but its influence on the combinatorial coefficients is taken into account. The two-loop self-energy subgraph of the second graph gives rise to a divergent at $\alpha = 0$ contribution to the vertex renormalisation constant Z_4.

Although this provides a clear indication of the inconsistency of the long-range scaling regime at small α, the actual borderline value α^* given by (6) is larger than that given by the exponent η at this order: $\alpha^* = \eta/2 = (n+2)\varepsilon^2/4(n+8)^2$. The reason is that not only the correction given by the three-loop graph of figure 1, but also similar self-energy graph insertions to all orders, should be taken into account to obtain a reliable value of α^* to the order ε^2, i.e. the one-loop vertex graph of figure 1 should be calculated with partially dressed propagators instead of the bare ones.

To this end, we note that the effect of each insertion of the self-energy subgraph of figure 1 is to multiply the original vertex graph by the factor $-a\lambda^2$, where

$$a = \frac{(n+2)\Gamma^2(2-2\alpha)\Gamma(1+\alpha)}{72\alpha(1-\alpha)\Gamma(3-3\alpha)} = \frac{n+2}{144\alpha} + \mathrm{O}(1) \tag{7}$$

and to shift the power of p^2 in the propagator with this insertion by ε, i.e. $p^{-2(1-\alpha)} \to p^{-2(1-\alpha+\varepsilon)}$. Let us denote

$$\int \frac{\mathrm{d}q}{(2\pi)^d} \frac{1}{q^{2u}(p-q)^{2v}} \equiv \frac{\Pi(u, v)}{p^{2(u+v-d/2)}}$$

where

$$\Pi(u, v) = \frac{\Gamma(d/2-u)\Gamma(d/2-v)\Gamma(u+v-d/2)}{(4\pi)^{d/2}\Gamma(u)\Gamma(v)\Gamma(d-u-v)}. \tag{8}$$

Then the account of all the insertions to the self-energy subgraph of figure 1 amounts to replacing the function $\Pi(1-\alpha, 1-\alpha)/p^{2(2-2\alpha-d/2)}$, which corresponds to the original one-loop vertex graph of figure 1, by the sum

$$\frac{\Pi(1-\alpha, 1-\alpha)}{p^{2(2-2\alpha-d/2)}} \to \sum_{k,l=0}^{\infty} b_{k,l}(-a\lambda^2)^{k+l} \frac{\Pi(1-\alpha+k\varepsilon, 1-\alpha+l\varepsilon)}{p^{2(2-2\alpha-d/2+(k+l)\varepsilon)}}$$

where $b_{k,l}$ are combinatorial coefficients. For the vertex renormalisation constant Z_4 we obtain

$$Z_4 = 1 + \frac{n+8}{6}\lambda \sum_{k,l}^{\infty} b_{k,l} \frac{(-a\lambda^2)^{k+l}}{(2k+2l+1)\varepsilon}$$

and the beta function takes the form

$$\beta_\lambda = \lambda\left(-\varepsilon + \frac{n+8}{6}\lambda \sum_{k,l}^{\infty} b_{k,l}(-a\lambda^2)^{k+l}\right). \tag{9}$$

The sum in this expression obviously corresponds to the sum of the original one-loop vertex graph and all vertex graphs produced from it by self-energy graph insertions of the type of figure 1, calculated for $\varepsilon = 0$, i.e. the sum on the right-hand side of (9) is equal to the convolution of two (partially) dressed propagators G calculated at the upper critical dimension. The expression for the dressed propagator G may by found by solving the truncated Dyson equation, depicted in figure 2, where the full lines

$G^{-1} = G_0^{-1} - \frac{1}{6}$ 

Figure 2. Truncated Dyson equation for the (partially) dressed propagator G. Full lines correspond to the propagator G and the vertices are those of the basic action (2) (in the full Dyson equation one of the vertices is also dressed).

correspond to the propagator G, and the vertices are those of the basic action (3). At the upper critical dimension the substitution $G(p)=A/p^{2(1-\alpha)}$, $G_0(p)=1/p^{2(1-\alpha)}$ transforms the integral equation of figure 2 to the following algebraic equation:

$$a\lambda^2A^4+A=1 \tag{10}$$

and since the sum on the right-hand side of the relation (9) is produced by a convolution of two dressed propagators, we can write

$$\beta_\lambda=\lambda\left(-\varepsilon+\frac{n+8}{6}\lambda A^2\right). \tag{11}$$

We are interested in the behaviour of the function A in the limit $\alpha\to 0$, for which we obtain from (7) and (10)

$$A=\frac{3^{1/2}2\alpha^{1/4}}{(n+2)^{1/4}\lambda^{1/2}}-\frac{3\alpha^{1/2}}{(n+2)^{1/2}\lambda}+\mathrm{O}\left[\left(\frac{\alpha}{\lambda^2}\right)^{3/4}\right]$$

which leads to the following asymptotic behaviour of the beta function (11):

$$\beta_\lambda=\lambda\left[-\varepsilon+\frac{2(n+8)\alpha^{1/2}}{(n+2)^{1/2}}-\frac{3^{1/2}2(n+8)\alpha^{3/4}}{(n+2)^{3/4}\lambda^{1/2}}+\mathrm{O}\left(\frac{\alpha}{\lambda}\right)\right].$$

From this expression it can be seen that the long-range regime, in fact, becomes inconsistent due to the fixed point ceasing to be a real number for small enough α. This occurs at the following value of $\alpha=\alpha^*$:

$$\alpha^*=\frac{(n+2)\varepsilon^2}{4(n+8)^2}$$

which coincides with the leading-order value obtained for the crossover from the short-range to the long-range regime [1].

4. Conclusion

To complete the analysis, we show that the critical exponent ν is also continuous at the crossover. The exponent ν is related to the anomalous dimension γ_{ϕ^2} of the 'mass operator' ϕ^2 as $\nu^{-1}=2-2\alpha-\gamma_{\phi^2}$, and is most conveniently calculated from the Green function $\Gamma_{\phi\phi,\phi^2}$, i.e. from the inverse propagator with one ϕ^2 insertion, and at the order $\mathrm{O}(\varepsilon)$ the coefficients of the ε expansion of both long-range and short-range models coincide in the limit $\alpha\to 0$. This result is not affected by the partial dressing of the propagators, since the contribution to the renormalisation constant Z_{ϕ_2} is given by a graph, which, apart from the index structure and combinatorial coefficient, coincides with the one-loop vertex graph of figure 1. Therefore, the contributions from the partial dressing of the propagators are the same in both cases and cancel at the fixed point. Further, the dressing of the propagators gives rise to an effective coupling constant λA^2, for which the fixed-point equation is the same as for λ in the original long-range model. Therefore, the continuity of the exponent ν may be checked to the order $\mathrm{O}(\varepsilon^2)$ by simply expanding in α the expressions obtained for the long-range model [7] and going to the limit $\alpha\to\alpha^*=(n+2)\varepsilon^2/4(n+8)^2$, as in the analysis of Sak [1].

In conclusion, we have shown that the crossover from the scaling regime described by the $\mathrm{O}(n)$ symmetric $(\phi^2)^2$ field theory with long-range exchange terms to the regime described by the corresponding short-range model can be consistently described in the framework of the field-theoretic renormalisation group, and at the crossover the critical exponents are continuous functions of the parameter α, which characterises the power-like fall-off of the long-range exchange term.

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